

# Axially Symmetric Shear-free Fluids in $f(R, T)$ Gravity

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## Abstract

In this work we have discussed the implications of shear-free condition on axially symmetric anisotropic gravitating objects in  $f(R, T)$  theory. Restricted axial symmetry ignoring rotation and reflection enteries is taken into account for establishment of instability range. Implementation of linear perturbation on constitutive modified dynamical equations yield evolution equation. This equation associates adiabatic index  $\Gamma$  with material and dark source components defining stable and unstable regions in Newtonian (N) and post-Newtonian (pN) approximations.

**Keywords:**  $f(R, T)$  gravity; Axial symmetry; Instability range; Shear-free Condition; Adiabatic index.

## 1 Introduction

In a recent work [1], we have discussed the instability range of axial system in  $f(R, T)$  gravity,  $R$  being Ricci scalar and  $T$  denoting trace of energy momentum. In continuation to [1], we plan to study the evolution of axially

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symmetric sources evolving under shear-free condition. Shear tensor has significant importance in the of structure formation, inhomogeneity factors and relativistic stellar phenomenons. Many authors [2]-[8] have discussed the impact of shear tensor in gravitational evolution and consequences of its vanishing value. It has been established recently [9] that the vanishing shear case has equivalence with the homology conditions and so has enormous significance in astrophysics [10, 11].

Possible aberrancies from most commonly studied spherical symmetry extended the concerns with non-spherical geometries of gravitating systems. Recent developments suggest that incidental variations in spherical symmetry prevails the more realistic scenarios. In this work, we have considered restricted axially symmetric system (ignoring reflection and rotation about symmetry axis) with anisotropic matter configuration evolving under shear-free condition. This study is conducted in  $f(R, T)$  gravity by taking into account a consistent  $f(R, T)$  model (satisfying  $\frac{df}{dr} \geq 0, \frac{d^2f}{dr^2} \geq 0$ ).

Study of gravitational collapse and its end-state has gained great deal of attention in past years. Dominance of inward acting gravity force over outward acting pressure leads to the gravitational collapse due to which star contracts to a point. A gravitating system remains stable until pressure keeps on balancing the gravitational pull and collapses when pressure to gravity balance fluctuates. The range of stability/instability primarily depends on mass, supersensitive stars lose stability more rapidly. Besides mass there are various factors that can modify instability range considerably such as shear, isotropy, anisotropy, radiation etc.

Instability problem received enormous attention after the seminal contribution of Chandrasekhar [12], he set stability criterion of gravitating sources in the form of adiabatic index  $\Gamma$  associating variation in pressure with varying energy density. The criterion for stability of gravitating systems with anisotropic matter distribution is established in [13]. Herrera and his collaborators [14]-[18] contributed greatly to study the dynamics of relativistic fluids to discuss the factors contributing in gravitational evolution. They also presented the general framework to deal with axially symmetric sources and their applications [19]-[21].

Gravitational interaction can be described in different manners, so far the scheme of general theory of relativity (GR) being a self-consistent approach has been largely used. It explains gravitational phenomenons adequately at cosmological scales [22]. However, developments in observational situations

[23]-[27] suggests that GR can not be the only theory that can be appropriate for all scales.

Many treatments have been made for suitable description of gravitational theories on large scales and deal with the issue of cosmic speed-up [28]-[37], by developing alternative gravitational theories [38]-[46] for e.g.  $f(R)$ ,  $f(R, T)$ ,  $f(G)$ ,  $f(T, \mathcal{T})$ , Brans-Dicke theory and so on. Such alternative theories are dynamically equivalent to GR at cosmological levels. Herein, we workout the gravitational evolution in  $f(R, T)$  theory introduced by Harko [47]. In this theory the matter content is considered to have interaction with the geometry. Many authors [48]-[50] have discussed the cosmological and thermodynamic implications in  $f(R, T)$  gravity.

The purpose of this work is to explore the implication of shear-free condition on dynamical instability of a restricted class of axially symmetric anisotropic systems in  $f(R, T)$  gravity. The gravitational action in  $f(R, T)$  includes an arbitrary function of  $R$  and  $T$ , written as [47]

$$\int dx^4 \sqrt{-g} \left[ \frac{f(R, T)}{16\pi G} + \mathcal{L}_{(m)} \right], \quad (1)$$

where  $\mathcal{L}_{(m)}$  is matter Lagrangian and  $g$  is the metric. A variety of choices of  $\mathcal{L}_{(m)}$  can be dealt with, each of which represent a specific configuration of relativistic matter.

The article is arranged as: Section 2 includes the discussion of considered spacetime, matter configuration, kinematic variables along with shear-free condition and modified dynamical equations. In section 3, we introduced the  $f(R, T)$  model and applied linear perturbation on conservation equations. Section 4 contains the discussion of weak field limit. Summary of the results is given in last section that is followed by an appendix.

## 2 Modified Dynamical Equations With Vanishing Shear

In this section, we develop the modified field equations in framework of  $f(R, T)$  gravity by considering axial system with anisotropic matter evolving under shear-free condition.

## 2.1 Metric and Matter Configuration

The dynamical systems lacking spherical symmetry have received great attention in recent past, since such abberations describes the more realistic phases of gravitational evolution. The general line element for axially symmetric gravitating sources constitutes five metric functions (independent), given by

$$ds^2 = -A^2(t, r, \theta)dt^2 + B^2(t, r, \theta)dr^2 + r^2B^2(t, r, \theta)d\theta^2 + C^2(t, r, \theta)d\phi^2 + 2G(t, r, \theta)dtd\theta + 2H(t, r, \theta)dtd\phi, \quad (2)$$

Herein, our goal is to study the implications of vanishing shear scalar on collapsing phenomenon. We have restricted the character of spacetime by imposing the highest degree of symmetry to avoid the complexities generated from the terms of reflection and rotation about symmetry axis. Thus, we have restricted the general line element (2) to somehow handle the problem under consideration and manipulate the results analytically. Considering vorticity-free case i.e. ignoring  $dtd\theta$  and  $dtd\phi$  terms in (2), the reduced form of line element containing three independent metric functions takes following form

$$ds^2 = -A^2(t, r, \theta)dt^2 + B^2(t, r, \theta)(dr^2 + r^2d\theta^2) + C^2(t, r, \theta)d\phi^2. \quad (3)$$

The matter distribution is taken to be anisotropic carrying unequal pressure stresses. The energy momentum tensor for usual matter is defined as [19]

$$T_{uv}^{(m)} = (\rho + p_\perp)V_uV_v - (K_uK_v - \frac{1}{3}h_{uv})(P_{zz} - P_{xx}) - (L_uL_v - \frac{1}{3}h_{uv})(P_{zz} - P_{yy}) + Pg_{uv} + 2K_{(u}L_{v)}P_{xy}, \quad (4)$$

where

$$P = \frac{1}{3}(P_{xx} + P_{yy} + P_{zz}), \quad h_{uv} = g_{uv} + V_uV_v,$$

where  $P_{xx}$ ,  $P_{yy}$ ,  $P_{zz}$  and  $P_{xy}$  denote corresponding anisotropic pressure stresses, satisfying  $P_{xy} = P_{yx}$  and  $P_{xx} \neq P_{yy} \neq P_{zz}$ . The four vectors in radial and axial directions are described by  $K_u$  and  $L_u$  respectively, while  $V_u$  stands for four-velocity and  $\rho$  is energy density. The above mentioned quantities are associated as

$$V_u = -A\delta_u^0, \quad K_u = B\delta_u^1, \quad L_u = rB\delta_u^2. \quad (5)$$

## 2.2 Kinematic Variables and Shear-free Condition

Kinematic variables play significantly important role in the description of self-gravitating sources. Herein three kinematic variables are contributing in the systems evolution, i.e., acceleration  $a_u$ , expansion scalar  $\Theta$  and the shear tensor  $\sigma_{uv}$ . These variables are obtained as follows

$$a_u = V^v V_{u;v} = (0, \frac{A'}{A}, \frac{A^\theta}{A}, 0), \quad (6)$$

$$\Theta = V^u_{;u} = \frac{1}{A} \left( \frac{2\dot{B}}{B} + \frac{\dot{C}}{C} \right), \quad (7)$$

$$\sigma_{uv} = V_{(u;v)} - a_{(u} V_{v)} - \frac{1}{3} \Theta (h_{uv}). \quad (8)$$

The non-zero components of  $\sigma_{uv}$  are

$$\sigma_{11} = \frac{B^2}{3A} \left( \frac{\dot{B}}{B} - \frac{\dot{C}}{C} \right), \quad (9)$$

$$\sigma_{22} = \frac{r^2 B^2}{3A} \left( \frac{\dot{B}}{B} - \frac{\dot{C}}{C} \right), \quad (10)$$

$$\sigma_{33} = \frac{-2C^2}{3A} \left( \frac{\dot{B}}{B} - \frac{\dot{C}}{C} \right). \quad (11)$$

The expression for shear scalar is obtained as [20]

$$\sigma = \frac{1}{A} \left( \frac{\dot{B}}{B} - \frac{\dot{C}}{C} \right). \quad (12)$$

Here dot and prime indicate derivatives w.r.t time and radial coordinates respectively. The shearing effects and inhomogeneity factors within the collapsing system play an essential role in the formation of the trapped surfaces and the apparent horizon. It is remarked in [6] that presence of shear causes distortion of trapped surface geometry which strongly implicates the development of naked singularities as end state of gravitational evolution. It is worth mentioning here that absence of shearing effects in collapsing cloud lead to the formation of a black hole. Thus, absence or presence of shear

have significant impression on kinematics of the gravitating source and its evolution.

The gravitating source under consideration is assumed to have vanishing shear, i.e.,  $\sigma = 0$ , that leads to  $\frac{\dot{B}}{B} = \frac{\dot{C}}{C}$ . We shall employ shear-free condition to study the dynamics of axially symmetric gravitating source.

### 2.3 Modified Field Equations

As mentioned earlier in introduction that different choices for matter Lagrangian can be taken to study the dynamics of a gravitating system, each choice represents a particular set of field equations. Here, we take  $\mathcal{L}_{(m)} = -\rho$ ,  $8\pi G = 1$  and vary the extended gravitational action (1) w.r.t metric  $g_{uv}$  as follows

$$G_{uv} = \frac{1}{f_R} \left[ (f_T + 1)T_{uv}^{(m)} + \rho g_{uv} f_T + \frac{f - Rf_R}{2} g_{uv} + (\nabla_u \nabla_v - g_{uv} \square) f_R \right], \quad (13)$$

where  $\square = \nabla^u \nabla_u$ ,  $f_R \equiv df(R, T)/dR$ ,  $f_T \equiv df(R, T)/dT$ ,  $\nabla_u$  stands for covariant derivative and  $T_{uv}^{(m)}$  denote the usual matter stress energy tensor. Corresponding components of effective Einstein tensor on account of shear-free condition are

$$G^{00} = \frac{1}{A^2 f_R} \rho + \frac{1}{A^2 f_R} \left[ \frac{f - Rf_R}{2} - \frac{3\dot{f}_R \dot{B}}{A^2 B} - \frac{f'_R}{B^2} \left( \frac{1}{r} + \frac{2B'}{B} - \frac{C'}{C} \right) - \frac{f_R^\theta}{r^2 B^2} \left( \frac{2B^\theta}{B} - \frac{C^\theta}{C} \right) + \frac{f_R''}{B^2} \right], \quad (14)$$

$$G^{01} = \frac{-1}{A^2 B^2 f_R} \left[ \frac{A'}{A} \dot{f}_R + \frac{\dot{B}}{B} f'_R - \dot{f}_R' \right], \quad (15)$$

$$G^{02} = \frac{-1}{r^2 A^2 B^2 f_R} \left[ \frac{A^\theta}{A} \dot{f}_R + \frac{\dot{B}}{B} f_R^\theta - \dot{f}_R^\theta \right], \quad (16)$$

$$G^{11} = \frac{1}{B^2 f_R} \left[ P_{xx}(f_T + 1) + \rho f_T - \frac{\dot{f}_R \dot{A}}{A^2 A} - \frac{f - R f_R}{2} - \frac{f_R^{\theta\theta}}{r^2 B^2} - \frac{\ddot{f}_R}{A^2} \right. \\ \left. + \frac{f'_R}{B^2} \left( \frac{1}{r} - \frac{A'}{A} + \frac{B'}{B} - \frac{C'}{C} \right) + \frac{f_R^\theta}{r^2 B^2} \left( \frac{3B^\theta}{B} - \frac{A^\theta}{A} - \frac{C^\theta}{C} \right) \right], \quad (17)$$

$$G^{12} = \frac{1}{r^2 B^4 f_R} \left[ P_{xy}(f_T + 1) + f_R'^\theta - \frac{B^\theta}{B} f'_R - \frac{B'}{B} f_R^\theta \right], \quad (18)$$

$$G^{22} = \frac{1}{r^2 B^4 f_R} \left[ P_{yy}(f_T + 1) + \rho f_T + \frac{\dot{f}_R}{A^2} \left( \frac{2\dot{B}}{B} - \frac{\dot{A}}{A} \right) + \frac{\ddot{f}_R}{A^2} - \frac{f - R f_R}{2} \right. \\ \left. - \frac{f_R''}{B^2} - \frac{f_R^\theta}{r^2 B^2} \left( \frac{A^\theta}{A} - \frac{B^\theta}{B} + \frac{C^\theta}{C} \right) - \frac{f'_R}{B^2} \left( \frac{A'}{A} - \frac{B'}{B} + \frac{C'}{C} \right) \right], \quad (19)$$

$$G^{33} = \frac{1}{C^2 f_R} \left[ P_{zz}(f_T + 1) + \frac{\ddot{f}_R}{A^2} - \frac{f_R^{\theta\theta}}{r^2 B^2} + \rho f_T - \frac{f - R f_R}{2} - \frac{\dot{f}_R}{A^2} \left( \frac{\dot{A}}{A} - \frac{2\dot{B}}{B} \right) \right. \\ \left. - \frac{f_R''}{B^2} - \frac{f'_R}{B^2} \left( \frac{A'}{A} - \frac{2B'}{B} - \frac{1}{r} \right) - \frac{f_R^\theta}{r^2 B^2} \left( \frac{A^\theta}{A} - \frac{2B^\theta}{B} \right) \right]. \quad (20)$$

The equation for Ricci scalar is found as

$$R = \frac{2}{A^2} \left[ \frac{3\dot{A}\dot{B}}{A B} - 3 \left( \frac{\dot{B}}{B} \right)^2 - \frac{2\ddot{B}}{B} - \frac{\ddot{C}}{C} \right] \\ + \frac{2}{B^2} \left[ \frac{A''}{A} + \frac{A'C'}{AC} + \frac{B''}{B} - \frac{1}{r} \left( \frac{A'}{A} - \frac{B'}{B} - \frac{C'}{C} \right) - \frac{B'^2}{B^2} \right. \\ \left. + \frac{C''}{C} + \frac{1}{r^2} \left( \frac{A^{\theta\theta}}{A} + \frac{B^{\theta\theta}}{B} + \frac{C^{\theta\theta}}{C} - \left( \frac{B^\theta}{B} \right)^2 + \frac{A^\theta C^\theta}{AC} \right) \right]. \quad (21)$$

## 2.4 Conservation Equations

In  $f(R, T)$  gravity, the divergence of energy momentum tensor leads to the following expression

$$\nabla^u T_{uv} = \frac{f_T(R, T)}{\kappa - f_T(R, T)} \left( [T_{uv} + \Theta_{uv}] \nabla^u \ln f_T(R, T) + \nabla^u \Theta_{uv} - \frac{1}{2} g_{\mu\nu} \nabla^\mu T \right). \quad (22)$$

It is clear from above equation that unlike GR the stress energy tensor remains non-conserved in  $f(R, T)$  gravity. This is because of the direct matter

geometry coupling, non-zero divergence induces a force acting orthogonally to four velocity leading to the non-geodesic path of test particles in non-minimally coupled gravitational theories. Dynamics of a system can be explored by considering conservation laws, divergence free gravitational theories can be dealt by taking conservation of energy momentum tensor. However, in our case we can not do so since  $T_{uv}$  is non-conserved that is why we take conservation of full form of field tensor i.e., effective Einstein tensor.

Bianchi identities are used here to arrive at conservation equations that are vital in establishment of evolution equation analytically. Following equations are obtained from Bianchi identities

$$G_{,0}^{00} + G_{,1}^{01} + G_{,2}^{02} + G^{00} \left( \frac{2\dot{A}}{A} + \frac{3\dot{B}}{B} \right) + G^{01} \left( \frac{3A'}{A} + \frac{2B'}{B} + \frac{C'}{C} + \frac{1}{r} \right) + G^{02} \left( \frac{3A^\theta}{A} + \frac{2B^\theta}{B} + \frac{C^\theta}{C} \right) + G^{11} \frac{B\dot{B}}{A^2} + G^{22} \frac{r^2 B\dot{B}}{A^2} + G^{33} \frac{C\dot{C}}{A^2} = 0, \quad (23)$$

$$G_{,0}^{01} + G_{,1}^{11} + G_{,2}^{12} + G^{00} \frac{AA'}{B^2} + G^{01} \left( \frac{\dot{A}}{A} + \frac{5\dot{B}}{B} \right) + G^{11} \left( \frac{A'}{A} + \frac{3B'}{B} + \frac{C'}{C} + \frac{1}{r} \right) + G^{12} \left( \frac{A^\theta}{A} + \frac{4B^\theta}{B} + \frac{C^\theta}{C} \right) - G^{22} \left( r + \frac{r^2 B'}{B} \right) + G^{33} \frac{CC'}{B^2} = 0, \quad (24)$$

$$G_{,0}^{02} + G_{,1}^{12} + G_{,2}^{22} + G^{00} \frac{AA^\theta}{r^2 B^2} + G^{02} \left( \frac{\dot{A}}{A} + \frac{5\dot{B}}{B} \right) - \frac{B^\theta}{r^2 B} G^{11} + \left( \frac{A'}{A} + \frac{4B'}{B} + \frac{C'}{C} + \frac{3}{r} \right) G^{12} + G^{22} \left( \frac{A^\theta}{A} + \frac{3B^\theta}{B} + \frac{C^\theta}{C} \right) - G^{33} \frac{CC^\theta}{r^2 B^2} = 0. \quad (25)$$

The notion 0, 1 and 2 stands for  $t, r$  and  $\theta$ . One can separate usual and dark source ingredients by substituting components of Einstein tensor from Eqs.(14)-(20) in Eqs.(23)-(25).

### 3 Perturbation Scheme

Theoretically and experimentally consistent  $f(R, T)$  models can be constructed, such models shall be compatible with local gravity tests and cosmological constraints. These constraints are required to meet for consistent solar system tests, matter domination phase and stable high-curvature configuration. Generally, a consistent gravitational model corresponds to the choice of parameters that are in accordance with the observational scenarios [51]. A



viable  $f(R, T)$  ghost-free model shall have positive first and second order derivatives for stability of cosmological perturbations [52]. In addition to this, any modified gravity model shall agree with weak field limit and must be stable at semiclassical and classical levels.

The viable models in  $f(R, T)$  gravity have classification based on fundamental form of  $f$ , as follows

- The model  $f(R, T) = R + 2f(T)$  describes the cosmological model accompanying time dependent and effective coupling, the term  $2f(T)$  characterizes the interaction of matter and curvature interaction in the extended gravitational action.
- $f(R, T) = f_1 + f_2$ ,  $f_1$  and  $f_2$  are arbitrary functions of  $R$  and  $T$ , respectively appearing explicitly.
- $f(R, T) = f_1(R) + f_2(R)f_3(T)$  denotes the most general class of  $f(R, T)$  models, carrying implicit form of  $R$  and  $T$  in extended gravitational action.

Herein, we are dealing with axial system analytically, in order to arrive at some fruitful conclusion avoiding complexities we select  $f(R, T) = f_1(R) + f_2(T)$  type. More particularly, we take  $f(R, T) = R + \alpha R^2 + \lambda T$ , where  $\alpha$  and  $\lambda$  denote positive constants. The origin of such constraints on model comes from the argument that non-linear terms of  $T$  in the model complicates the field equations that can not be tackled by analytic approach. The  $f(R, T)$  models containing non-linear enteries of  $T$  can be handled by numerical simulations but such analysis provide outcomes for a specific model. Whereas, more generic results can be found for a variety of models with arbitrary values of  $\alpha$  and  $\lambda$  by using analytic approach. .

Following scheme for perturbation of physical variables is employed to monitor the fluctuations in the gravitating object with the time transition. Metric coefficients and Ricci scalar are perturbed by using following pattern

$$L(t, r, \theta) = L_0(r, \theta) + \epsilon D(t)l(r, \theta). \quad (26)$$

While energy density and pressures stresses are perturbed as follows

$$\rho(t, r, \theta) = \rho_0(r, \theta) + \epsilon \bar{\rho}(t, r, \theta), \quad (27)$$

$$P_{ij}(t, r, \theta) = P_{ij0}(r, \theta) + \epsilon \bar{P}_{ij}(t, r, \theta). \quad (28)$$

Variation in  $f(R, T)$  model is taken in the following form

$$f(R, T) = [R_0(r, \theta) + \alpha R_0^2(r, \theta) + \lambda T_0(r, \theta)] + \epsilon D(t) e(r, \theta) [1 + 2\alpha R_0(r, \theta)], \quad (29)$$

$$f_R = 1 + 2\alpha R_0(r, \theta) + \epsilon 2\alpha D(t) e(r, \theta), \quad (30)$$

$$f_T = \lambda, \quad (31)$$

where  $0 < \epsilon \ll 1$ . Application of the first order perturbation on dynamical equations (23)-(25) implies

$$\left[ \dot{\bar{\rho}} + \left\{ \rho_0 \left( \frac{a}{A_0} + \frac{3\lambda_1 b}{B_0} \right) + \frac{\lambda_1 b}{B_0} (P_{xx0} + P_{yy0} + P_{zz0}) + Z_{1p} \right\} \dot{D} \right] = 0, \quad (32)$$

$$\begin{aligned} & \left[ \lambda_1 \bar{P}_{xx} + \lambda \bar{\rho} - 2(\lambda_1 P_{xx0} + \lambda \rho_0) \left( \frac{b}{B_0} + \frac{e\alpha}{I} \right) D \right]_{,1} + (\lambda_1 \bar{P}_{xx} + \lambda \bar{\rho}) \left( \frac{A'_0}{A_0} + \frac{3B'_0}{B_0} \right. \\ & \left. + \frac{C'_0}{C_0} + \frac{1}{r} \right) + \frac{1}{r^2} \left[ \lambda_1 \bar{P}_{xy} - 2 \left( \frac{2b}{B_0} + \frac{e\alpha}{I} \right) P_{xy0} D \right]_{,2} + \frac{\lambda_1 \bar{P}_{xy}}{r^2 B_0^2} \left( \frac{A_0^\theta}{A_0} + 4 \frac{B_0^\theta}{B_0} + \frac{C_0^\theta}{C_0} \right) + \\ & (\lambda_1 \bar{P}_{yy} + \lambda \bar{\rho}) \left( \frac{1}{r} + \frac{B'_0}{B_0} \right) + (\lambda_1 \bar{P}_{zz} + \lambda \bar{\rho}) \frac{C'_0}{C_0} + D \left[ (\lambda_1 P_{xx0} + \lambda \rho_0) \left( \left( \frac{a}{A_0} \right)' + \right. \right. \\ & \left. \left. + 4 \left( \frac{b}{B_0} \right)' - \left( \frac{2b}{B_0} + \frac{e\alpha}{I} \right) \left( \frac{A'_0}{A_0} + \frac{3B'_0}{B_0} + \frac{C'_0}{C_0} + \frac{1}{r} \right) \right) + (\lambda_1 P_{yy0} + \lambda \rho_0) \left( \left( \frac{b}{B_0} \right)' \right. \right. \\ & \left. \left. - \left( \frac{2b}{B_0} + \frac{e\alpha}{I} \right) \frac{B'_0}{B_0} \right) \left( \frac{1}{r} + \frac{B'_0}{B_0} \right) + (\lambda_1 P_{zz0} + \lambda \rho_0) \left( \left( \frac{c}{C_0} \right)' - \left( \frac{2b}{B_0} + \frac{e\alpha}{I} \right) \frac{C'_0}{C_0} \right) \right. \\ & \left. + \lambda_1 P_{xy0} \left( \left( \frac{a}{A_0} \right)^\theta + 5 \left( \frac{b}{B_0} \right)^\theta - \left( \frac{2b}{B_0} + \frac{e\alpha}{I} \right) \frac{C_0^\theta}{C_0} \right) \right] + Z_{2p} = 0, \quad (33) \end{aligned}$$

$$\begin{aligned}
& \left[ \lambda_1 \bar{P}_{yy} + \lambda \bar{\rho} - 2(\lambda_1 P_{yy0} + \lambda \rho_0) \left( \frac{b}{B_0} + \frac{e\alpha}{I} \right) D \right]_{,2} + \left[ \frac{1}{r^2 B_0^4 I} \lambda_1 \bar{P}_{xy} \right]' + \bar{\rho} \frac{A_0^\theta}{A_0} \\
& + (\lambda_1 \bar{P}_{xx} + \lambda \bar{\rho}) \frac{B_0^\theta}{B_0} + \lambda_1 \bar{P}_{xy} \left( \frac{A_0'}{A_0} + \frac{4B_0'}{B_0} + \frac{C_0'}{C_0} + \frac{3}{r} \right) + (\lambda_1 \bar{P}_{yy} + \lambda \bar{\rho}) \left( \frac{A_0^\theta}{A_0} \right. \\
& + 3 \frac{B_0^\theta}{B_0} + \frac{C_0^\theta}{C_0} \left. \right) + (\lambda_1 \bar{P}_{zz} + \lambda \bar{\rho}) \frac{C_0^\theta}{C_0} + D \left[ \rho_0 \left( \left( \frac{a}{A_0} \right)^\theta - 2 \left( \frac{b}{B_0} + \frac{e\alpha}{I} \right) \frac{A_0^\theta}{A_0} \right) \right. \\
& + \lambda_1 P_{xy0} \left( \left( \frac{a}{A_0} \right)' + 5 \left( \frac{b}{B_0} \right)' - \left( \frac{4b}{B_0} + \frac{2e\alpha}{I} \right) \left( \frac{A_0'}{A_0} + \frac{4B_0'}{B_0} + \frac{C_0'}{C_0} \right. \right. \\
& + \left. \left. \frac{3}{r} \right) \right) + (\lambda_1 P_{xx0} + \lambda \rho_0) \left( 4 \left( \frac{b}{B_0} \right)^\theta - \left( \frac{2b}{B_0} + \frac{e\alpha}{I} \right) \frac{B_0^\theta}{B_0} \right) + (\lambda_1 P_{yy0} + \lambda \rho_0) \\
& \times \left( \left( \frac{a}{A_0} \right)^\theta + 4 \left( \frac{b}{B_0} \right)^\theta - \left( \frac{2b}{B_0} + \frac{e\alpha}{I} \right) \left( \frac{A_0^\theta}{A_0} + 3 \frac{B_0^\theta}{B_0} + \frac{C_0^\theta}{C_0} \right) \right) \\
& + (\lambda_1 P_{zz0} + \lambda \rho_0) \left( \left( \frac{b}{B_0} \right)^\theta - \left( \frac{2b}{B_0} + \frac{e\alpha}{I} \right) \frac{C_0^\theta}{C_0} \right) \left. \right] + Z_{3p} = 0, \tag{34}
\end{aligned}$$

where expressions for  $Z_{1p}$ ,  $Z_{2p}$  and  $Z_{3p}$  given in appendix. We take,  $I = 1 + 2\alpha R_0$  and  $J = e2\alpha R_0$  for the sake of simplicity. The value of energy density  $\bar{\rho}$  is derived from Eq.(32) which turns out to be

$$\bar{\rho} = - \left\{ \rho_0 \left( \frac{a}{A_0} + \frac{3\lambda_1 b}{B_0} \right) + \frac{\lambda_1 b}{B_0} (P_{xx0} + P_{yy0} + P_{zz0}) + Z_{1p} \right\} D. \tag{35}$$

The expression for  $\rho$  and pressure stresses can be related as [1]

$$\bar{P}_i = \Gamma \frac{p_{i0}}{\rho_0 + p_{i0}} \bar{\rho}. \tag{36}$$

where  $\Gamma$  represents the variation of anisotropic pressure stresses with the varying energy density and index  $i = xx, yy, xy, zz$ . Making use of Eq.(36) in Eq.(35) and some algebraic manipulations provide linearly perturbed anisotropic stresses. An ordinary differential equation is obtained by using linearly perturbed form of Ricci scalar having solution as follows

$$D(t) = -e^{\sqrt{Z_4}t}. \tag{37}$$

The expression for  $Z_4$  is given in appendix, Eq.(37) is holds for positive values of  $Z_4$ .

## 4 Weak Field Limit

In this section, we found the entries belonging to Newtonian (N) and post Newtonian (pN) eras. Equations (37) and (36) are inserted in Eq. (33) to arrive at evolution equation from which instability criterion is developed in terms of adiabatic index.

### 4.1 N Approximation

In this regime, we take  $A_0 = 1$ ,  $B_0 = 1$ ,  $\rho_0 \gg p_{i0}$  and Schwarzschild coordinates  $C_0 = r$  in evolution equation, whose outcome reveals the following relation

$$\Gamma < \frac{\lambda N'_0 - \frac{3}{r}N_0 - 2\lambda(\rho_0 N_2)' - \frac{2}{r}(P_{xy0}N_2)^\theta + \lambda N_2 N_3 - \frac{2}{r}N_2 + \lambda P_{xy0}N_4 + Z_{2_p}^N}{\lambda_1(P_{xx0}N_1)' + \frac{\lambda_1}{r^2}(P_{xy0}N_1)^\theta - \frac{1}{r}N_1(P_{xx0} + P_{yy0} + P_{zz0})}, \quad (38)$$

where  $Z_{2_p}^N$  represent those terms of  $Z_{2p}$  that belongs to the N-limit. Furthermore

$$\begin{aligned} N_0 &= - \left\{ \rho_0 N_1 + \frac{\lambda_1 c}{r}(P_{xx0} + P_{yy0} + P_{zz0}) + Z_{1_p}^N \right\}, \\ N_1 &= a + \frac{3\lambda_1 c}{r}, \quad N_2 = \frac{c}{r} + \frac{\alpha e}{I} \\ N_3 &= a' + 4b' + 2\left(\frac{c}{r}\right)', \quad N_4 = a^\theta + 4b^\theta + \frac{c^\theta}{r}. \end{aligned}$$

Both usual matter and dark source entries are taking part in above inequality for  $\Gamma$ , gravitational sources maintain stability for those values of physical parameters for which the inequality (38) is satisfied. All entries appearing in (38) shall remain positive, to fulfil this requirement we need to constrain some of the physical parameters. The restrictions in N-limit are

$$P_{xx0} + P_{yy0} + P_{zz0} < \frac{\lambda_1 r}{N_1}((P_{xx0}N_1)' + \frac{1}{r^2}), \quad (P_{xy0}N_2)^\theta < -2\lambda(\rho_0 N_2)'.$$

Stability of the gravitating system distorts whenever above constraints are violated that leads to the instability range of collapsing stars.

## 4.2 pN Approximation

In pN limit, we take  $A_0 = 1 - \frac{m_0}{r}$  and  $B_0 = 1 + \frac{m_0}{r}$ , use of these assumptions in evolution equation leads to following inequality for  $\Gamma$

$$\Gamma < \frac{\lambda N'_{10} + N_9 N_{10} - 2\lambda(\rho_0 N_6)' - \frac{2}{r}(P_{xy0} N_6)^\theta + \lambda \rho_0 N_7 - \frac{3}{r} N_6 + \lambda P_{xy0} N_8 + Z_{2_p^{pN}}}{\lambda_1 (P_{xx0} N_5)' + \frac{\lambda_1}{r^2} (P_{xy0} N_5)^\theta - \frac{1}{r} N_5 (P_{xx0} + P_{yy0} + P_{zz0}) + N_{11}}, \quad (39)$$

where

$$\begin{aligned} N_5 &= \left( \frac{ar}{r - m_0} + \frac{2\lambda_1 br}{r + m_0} + \frac{\lambda_1 c}{r} \right), \quad N_6 = \frac{2\lambda_1 br}{r + m_0} + \frac{e\alpha}{I}, \\ N_7 &= \left( \frac{ar}{r - m_0} \right)' + 4 \left( \frac{br}{r + m_0} \right)' + \left( \frac{2c}{r} \right)' - N_6 \left( \frac{2}{r} + \left( \frac{m_0}{r} \right)' \frac{3r}{r + m_0} \right), \\ N_8 &= \left( \frac{ar}{r - m_0} \right)^\theta + \left( \frac{br}{r + m_0} \right)^\theta + \left( \frac{c}{r} \right)^\theta, \quad N_9 = \left( \frac{3}{r} + \left( \frac{m_0}{r} \right)' \frac{3r}{r + m_0} \right), \\ N_{10} &= - \left\{ \rho_0 N_5 + \frac{2\lambda_1 br}{r + m_0} (P_{xx0} + P_{yy0}) + \frac{\lambda_1 c}{r} P_{zz0} + Z_{1_p^{pN}} \right\}, \\ N_{11} &= \frac{P_{xy0} N_5}{(r + m_0)^2} \left( \left( \frac{ar}{r - m_0} \right)^\theta + \left( \frac{4br}{r + m_0} \right)^\theta \right). \end{aligned}$$

Likewise N-limit the restrictions in metric functions and dark source entries are required to achieve stability of gravitating sources. System is exposed to gravitational collapse whenever the inequality (39) breaks down. Some of the constraints are

$$\begin{aligned} P_{xy} \left[ \frac{2\lambda_1 br}{r + m_0} + \frac{e\alpha}{I} \right] &< 0, \\ \left\{ \rho_0 N_5 + \frac{2\lambda_1 br}{r + m_0} (P_{xx0} + P_{yy0}) + \frac{\lambda_1 c}{r} P_{zz0} + Z_{1_p^{pN}} \right\} &< 0. \end{aligned}$$

## 5 Summary

Study of non-spherical symmetries provide deep insight of more realistic settings such as weak lensing, large scale structures, Planck data, cosmic microwave background etc. In this work, we have studied the impact of restricted axial symmetry (ignoring reflection and rotation) on anisotropic

$f(R, T)$  gravity model evolving under shear-free condition. Restricted character of spacetime incorporates three independent metric functions depending on  $t$ ,  $r$  and  $\theta$ , that leads to the vorticity-free case. We choose,  $f(R, T) = R + \alpha R^2 + \lambda T$  with positive values of  $\alpha$  and  $\lambda$  as a viable  $f(R, T)$  model to discuss the instability range.

The variation of extended gravitational action (1) is considered to obtain the modified field equations in  $f(R, T)$  gravity. We make use of shear-free condition to evaluate components of modified field equations. The energy momentum tensor remains non-conserved in  $f(R, T)$  gravity that is why we take conservation of modified Einstein tensor. The conservation equations lead to highly non-linear complicated equations. To tackle with the complexities of dynamical equations, we implement linear perturbation on all physical parameters. Application of linear perturbation yield equations for perturbed energy density that is further used to evaluate perturbed anisotropic stresses.

Insertion of perturbed stresses and energy density in second Bianchi identity leads to the collapse equation carrying adiabatic index that elaborates the variation of anisotropic pressures with energy density. It is found that imposed shear-free condition reduces the entries with negative sign in collapse equation that leads to less restricted and enhanced regions of stability. Thus it can be remarked that vanishing shear somehow delays the collapsing phenomenon. Moreover, weak field limit is checked by evaluating expression for adiabatic index in Newtonian and post Newtonian approximations. Corrections to GR can be recovered by taking  $\alpha \rightarrow 0, \lambda \rightarrow 0$  in evolution equation. While  $\lambda \rightarrow 0$  reduces results in  $f(R)$  gravity for Starobinsky model.

## Appendix

$$\begin{aligned}
Z_{1p} = & \frac{e}{2} - A_0^2 \left\{ \frac{1}{A_0^2 B_0^2 I^2} \left( (2\alpha e R_0)' \left( 1 - \frac{b}{B_0} \right) - 2\alpha e R_0 \frac{A_0'}{A_0} \right) \right\}_{,1} - \frac{A_0^2}{r^2} \left\{ \frac{2}{A_0^2 B_0^2 I^2} \right. \\
& \times \left( (\alpha e R_0)^\theta \left( 1 - \frac{b}{B_0} \right) - (\alpha e R_0) \frac{A_0^\theta}{A_0} \right) \left. \right\}_{,2} + \frac{\alpha^2 R_0^3}{I} + \frac{1}{B_0^2} \left[ \frac{(e^\theta (2\alpha e R_0))^\theta}{r^2} - 4\alpha \right. \\
& \times \left( (R_0 R_0')' + \frac{(R_0 R_0^\theta)^\theta}{r^2} \right) \left( \frac{a}{A_0} + \frac{b}{B_0} + \frac{\alpha e R_0}{I} \right) + I' \left\{ - \left( \frac{b}{B_0} \right)' \right. \\
& - \frac{b}{B_0} \left( \frac{3A_0'}{A_0} + \frac{2B_0'}{B_0} - \frac{C_0'}{C_0} - \frac{4}{r} \right) - \frac{c}{C_0} \left( \frac{A_0'}{A_0} - \frac{C_0'}{C_0} - \frac{1}{r} \right) + \frac{(2\alpha e R_0)}{I} \left( \frac{C_0'}{C_0} + \frac{2B_0'}{B_0} \right. \\
& \left. \left. - \frac{3}{r} \right) \right\} + (e'(2\alpha e R_0))' + \frac{I^\theta}{r^2} \left\{ - \left( \frac{b}{B_0} \right)^\theta - \frac{b}{B_0} \left( \frac{3A_0^\theta}{A_0} + \frac{2B_0^\theta}{B_0} - \frac{C_0^\theta}{C_0} \right) \right. \\
& + \frac{(2\alpha e R_0)}{I} \left( \frac{C_0^\theta}{C_0} + \frac{2B_0^\theta}{B_0} \right) \left. \right\} + (2\alpha e R_0)' \left( \frac{C_0'}{C_0} - \frac{2B_0'}{B_0} + \frac{1}{r} \right) \\
& + \frac{(2\alpha e R_0)^\theta}{r^2} \left( \frac{C_0^\theta}{C_0} - \frac{2B_0^\theta}{B_0} \right) + \left( \frac{2a}{A_0} + \frac{b}{B_0} \right) \left( I'' + \frac{I^{\theta\theta}}{r^2} \right) - \left( \frac{3A_0'}{A_0} + \frac{2B_0'}{B_0} + \frac{1}{r} \right. \\
& + \frac{C_0'}{C_0} \left. \right) \left( \frac{(2\alpha e R_0)'}{I} \left( 1 - \frac{b}{B_0} \right) - \frac{A_0'}{A_0} \frac{(2\alpha e R_0)}{I} \right) + \left( \frac{A_0^\theta (2\alpha e R_0)}{A_0 I} - \frac{(2\alpha e R_0)^\theta}{I} \left( 1 - \frac{b}{B_0} \right) \right) \\
& \left. \left( \frac{3A_0^\theta}{A_0} + \frac{C_0^\theta}{C_0} + \frac{2B_0^\theta}{B_0} \right) \right], \tag{40}
\end{aligned}$$

$$\begin{aligned}
Z_{2p} = & \left[ \left[ \frac{1}{IB_0^2} \left\{ \frac{\ddot{D}}{DA_0^2} - \frac{1}{B_0^2} \left\{ \frac{eB_0^2}{2} + I' \left( \frac{a}{A_0} \right)' \right. \right. \right. \right. \\
& + \left( J' - \frac{2b}{B_0} I' \right) \left( \frac{A'_0}{A_0} + \frac{C'_0}{C_0} - \frac{B'_0}{B_0} - \frac{1}{r} \right) + \frac{1}{r^2} \left( J^{\theta\theta} + \left( J^\theta - \frac{2b}{B_0} I^\theta \right) \left( \frac{A_0^\theta}{A_0} \right. \right. \\
& \left. \left. \left. - \frac{3B_0^\theta}{B_0} + \frac{C_0^\theta}{C_0} \right) + \frac{2b}{B_0} I^{\theta\theta} + I^\theta \left( \left( \frac{a}{A_0} \right)^\theta - 2 \left( \frac{b}{B_0} \right)^\theta \right) \right) \right\} \right\} \right]_{,1} \\
& + \left[ \frac{1}{r^2 IB_0^4} \left\{ J^{\theta\theta} + \left( \frac{b}{B_0} \right)^\theta I' + J^\theta \left( \frac{B'_0}{B_0} + \frac{1}{r} \right) - \left( \frac{b}{B_0} \right)' I^\theta \right\} \right]_{,2} IB_0^4 \\
& - e \frac{B'_0}{B_0} + \frac{A'_0}{A_0} \left[ J'' + \frac{J^{\theta\theta}}{r^2} - \frac{2b}{B_0} \left( I'' + \frac{I^{\theta\theta}}{r^2} \right) + \left( J' - \frac{2b}{B_0} I' \right) \left( \frac{C'_0}{C_0} - \frac{2B'_0}{B_0} \right. \right. \\
& \left. \left. + \frac{1}{r} \right) - \frac{1}{r^2} \left\{ I^\theta \left( \frac{b}{B_0} \right)^\theta - (J^\theta \right. \right. \\
& \left. \left. - \frac{2b}{B_0} I^\theta \right) \left( \frac{C_0^\theta}{C_0} - \frac{2B_0^\theta}{B_0} \right) \right\} \right] + \left( \frac{(aA_0)'}{A_0^2} - \frac{2b}{B_0} \frac{A'_0}{A_0} \right) \left( \frac{\alpha R_0^2 B_0^2}{2} + I'' + I' \left( \frac{C'_0}{C_0} \right. \right. \\
& \left. \left. - \frac{2B'_0}{B_0} + \frac{1}{r} \right) + \frac{I^{\theta\theta}}{r^2} + \frac{I^\theta}{r^2} \left( \frac{C_0^\theta}{C_0} - \frac{2B_0^\theta}{B_0} \right) \right) - \left\{ \frac{\alpha R_0^2 B_0^2}{2} + I' \left( \frac{A'_0}{A_0} + \frac{C'_0}{C_0} - \frac{B'_0}{B_0} \right. \right. \\
& \left. \left. - \frac{1}{r} \right) + \frac{I^{\theta\theta}}{r^2} + \frac{I^\theta}{r^2} \left( \frac{A_0^\theta}{A_0} + \frac{C_0^\theta}{C_0} - \frac{3B_0^\theta}{B_0} \right) \right\} \left( \left( \frac{a}{A_0} \right)' + 4 \left( \frac{b}{B_0} \right)' \right) \\
& - \left( \frac{A'_0}{A_0} + \frac{C'_0}{C_0} + \frac{3B'_0}{B_0} + \frac{1}{r} \right) \left\{ I' \left( \frac{a}{A_0} \right)' + \left( \frac{A'_0}{A_0} - \frac{B'_0}{B_0} \right. \right. \\
& \left. \left. + \frac{C'_0}{C_0} - \frac{1}{r} \right) \left( J' - \frac{2b}{B_0} I' \right) + \frac{1}{r^2} \left( J^{\theta\theta} + \left( J^\theta - \frac{2b}{B_0} I^\theta \right) \left( \frac{A_0^\theta}{A_0} - \frac{3B_0^\theta}{B_0} + \frac{C_0^\theta}{C_0} \right) \right. \right. \\
& \left. \left. + \frac{2b}{B_0} I^{\theta\theta} + I^\theta \left( \left( \frac{a}{A_0} \right)^\theta - 2 \left( \frac{b}{B_0} \right)^\theta \right) \right) \right\} - \left[ \left( \left( \frac{a}{A_0} \right)^\theta \right. \right. \\
& \left. \left. + 5 \left( \frac{b}{B_0} \right)^\theta \right) \left( I^{\theta\theta} + \frac{B_0^\theta}{B_0} I' + I^\theta \left( \frac{B'_0}{B_0} + \frac{1}{r} \right) \right) - \left( \frac{A_0^\theta}{A_0} + \frac{4B_0^\theta}{B_0} + \frac{C_0^\theta}{C_0} \right) \left( \frac{B_0^\theta}{B_0} J' \right. \right. \\
& \left. \left. - J^{\theta\theta} - I' \left( \frac{b}{B_0} \right)^\theta - J^\theta \left( \frac{B'_0}{B_0} + \frac{1}{r} \right) + I^\theta \left( \frac{b}{B_0} \right)' \right) \right] \frac{1}{r^2} - \left( \frac{B'_0}{B_0} + \frac{1}{r} \right) \left[ \frac{B_0^2}{A_0^2} \frac{\ddot{D}}{D} J \right. \\
& \left. - J'' + \frac{2b}{B_0} I'' - I' \left( \frac{a}{A_0} \right)' - \left( J' - \frac{2b}{B_0} I' \right) \left( \frac{A'_0}{A_0} + \frac{C'_0}{C_0} \right. \right.
\end{aligned}$$



$$\begin{aligned}
& -\frac{B'_0}{B_0} \Big) + \frac{1}{r^2} \left( I^\theta \left( \frac{a}{A_0} \right)^\theta - \left( J^\theta - \frac{2b}{B_0} I^\theta \right) \left( \frac{A_0^\theta}{A_0} - \frac{B_0^\theta}{B_0} \right. \right. \\
& \left. \left. + \frac{C_0^\theta}{C_0} \right) \right) \Big) + \left( \frac{b}{B_0} \right)' \left[ \frac{LB_0^2}{2} + I' \left( \frac{A'_0}{A_0} + \frac{C'_0}{C_0} - \frac{B'_0}{B_0} \right) - \frac{I^\theta}{r^2} \left( \frac{A_0^\theta}{A_0} - \frac{B_0^\theta}{B_0} + \frac{C_0^\theta}{C_0} \right) \right. \\
& \left. - I'' \right] + \frac{C'_0}{C_0} \left[ J'' - \frac{2b}{B_0} I'' + I' \left( \left( \frac{a}{A_0} \right)' - \left( \frac{2b}{B_0} \right)' \right) + \left( J' - \frac{2b}{B_0} I' \right) \left( \frac{A'_0}{A_0} - \frac{B'_0}{B_0} \right. \right. \\
& \left. \left. + \frac{1}{r} \right) + \frac{1}{r^2} \left\{ J^{\theta\theta} + \left( J^\theta - \frac{2b}{B_0} I^\theta \right) \left( \frac{A_0^\theta}{A_0} - \frac{2B_0^\theta}{B_0} \right) + I^\theta \left( \left( \frac{a}{A_0} \right)^\theta - 2 \left( \frac{b}{B_0} \right)^\theta \right) \right. \right. \\
& \left. \left. - \frac{2b}{B_0} I^{\theta\theta} \right\} \right] + \left( \frac{(cC_0)'}{C_0^2} - \frac{2b}{B_0} \frac{C'_0}{C_0} \right) \left[ I' \left( \frac{A'_0}{A_0} - \frac{2B'_0}{B_0} + \frac{1}{r} \right) - \frac{I^\theta}{r^2} \left( \frac{A_0^\theta}{A_0} - \frac{2B_0^\theta}{B_0} \right) \right. \\
& \left. + \frac{\alpha R_0^2 B_0^2}{2} + I'' + \frac{I^{\theta\theta}}{r^2} \right] - \frac{\ddot{D}B_0^2}{DA_0^2 I} \left( J' - \frac{A'_0}{A_0} J - \frac{b}{B_0} I' \right), \tag{41}
\end{aligned}$$

$$\begin{aligned}
Z_{3p} &= Ir^2 B_0^4 \left[ \left[ \frac{1}{r^2 I B_0^4} \left\{ J^\theta + \left( \frac{b}{B_0} \right)^\theta I' + J^\theta \left( \frac{B'_0}{B_0} + \frac{1}{r} \right) - \left( \frac{b}{B_0} \right)' I^\theta \right\} \right]_{,1} \right. \\
&+ \frac{\ddot{D}B_0^2}{DA_0^2 I} \left( \frac{A_0^\theta}{A_0} J + \frac{b}{B_0} I^\theta - J^\theta \right) + \left[ \frac{1}{Ir^2 B_0^4} \left\{ \frac{\ddot{D}B_0^2}{DA_0^2} J - J'' + \frac{2b}{B_0} I'' + \left( \frac{2b}{B_0} I' \right. \right. \right. \\
&- J' \left. \left. \right) \left( \frac{A'_0}{A_0} + \frac{C'_0}{C_0} - \frac{B'_0}{B_0} \right) - I' \left( \frac{a}{A_0} \right)' + \frac{1}{r^2} \left( \left( \frac{2b}{B_0} I^\theta \right. \right. \right. \\
&- J^\theta \left. \left. \right) \left( \frac{A_0^\theta}{A_0} - \frac{B_0^\theta}{B_0} + \frac{C_0^\theta}{C_0} \right) - I^\theta \left( \frac{a}{A_0} \right)^\theta \right) \left. \right\} \right]_{,2} \Big] \\
&- e \frac{B_0^\theta}{B_0} + \frac{A_0^\theta}{A_0} \left[ J'' + \frac{J^{\theta\theta}}{r^2} - \frac{2b}{B_0} \left( I'' + \frac{I^{\theta\theta}}{r^2} \right) + \left( J' - \frac{2b}{B_0} I' \right) \left( \frac{C'_0}{C_0} - \frac{2B'_0}{B_0} \right. \right. \\
&+ \frac{1}{r} \left. \right) - \frac{1}{r^2} \left\{ I^\theta \left( \frac{b}{B_0} \right)^\theta - \left( J^\theta \right. \right. \\
&- \frac{2b}{B_0} I^\theta \left. \left. \right) \left( \frac{C_0^\theta}{C_0} - \frac{2B_0^\theta}{B_0} \right) \right\} \Big] + \left( \frac{(aA_0)^\theta}{A_0^2} - \frac{2b}{B_0} \frac{A_0^\theta}{A_0} \right) \left( \frac{\alpha R_0^2 B_0^2}{2} + I'' + I' \left( \frac{C'_0}{C_0} \right. \right.
\end{aligned}$$

$$\begin{aligned}
& -\frac{2B'_0}{B_0} + \frac{1}{r} \Big) + \frac{I^{\theta\theta}}{r^2} + \frac{I^\theta}{r^2} \left( \frac{C_0^\theta}{C_0} - \frac{2B_0^\theta}{B_0} \right) \Big) - \left( \frac{b}{B_0} \right)^\theta \left\{ \frac{\alpha R_0^2 B_0^2}{2} + I' \left( \frac{A'_0}{A_0} + \frac{C'_0}{C_0} \right. \right. \\
& \left. \left. - \frac{B'_0}{B_0} + \frac{1}{r} \right) + \frac{I^{\theta\theta}}{r^2} + \frac{I^\theta}{r^2} \left( \frac{A_0^\theta}{A_0} + \frac{C_0^\theta}{C_0} - \frac{3B_0^\theta}{B_0} \right) \right\} - \frac{B_0^\theta}{B_0} \left\{ I' \left( \frac{a}{A_0} \right)' \right. \\
& \left. + \left( \frac{A'_0}{A_0} - \frac{B'_0}{B_0} + \frac{C'_0}{C_0} - \frac{1}{r} \right) \left( J' - \frac{2b}{B_0} I' \right) + \frac{1}{r^2} \left( J^{\theta\theta} - \left( \frac{2b}{B_0} I^\theta - \right. \right. \right. \\
& \left. \left. \left. J^\theta \right) \left( \frac{A_0^\theta}{A_0} - \frac{3B_0^\theta}{B_0} + \frac{C_0^\theta}{C_0} \right) + \frac{2b}{B_0} I^{\theta\theta} + I^\theta \left( \left( \frac{a}{A_0} \right)^\theta - 2 \left( \frac{b}{B_0} \right)^\theta \right) \right) \right\} \\
& - \frac{1}{r^2} \left[ \left( \left( \frac{a}{A_0} \right)' + 2 \left( \frac{b}{B_0} \right)' \right) \left( I^{\theta\theta} + \frac{B_0^\theta}{B_0} I' + I^\theta \left( \frac{B'_0}{B_0} + \frac{1}{r} \right) \right) - \left( \frac{A'_0}{A_0} \right. \right. \\
& \left. \left. + \frac{4B'_0}{B_0} + \frac{C'_0}{C_0} \right) \left( \frac{B_0^\theta}{B_0} J' - J^\theta - I' \left( \frac{b}{B_0} \right)^\theta - J^\theta \left( \frac{B'_0}{B_0} + \frac{1}{r} \right) + I^\theta \left( \frac{b}{B_0} \right)' \right) \right] \\
& + \left( \frac{A_0^\theta}{A_0} + \frac{3B_0^\theta}{B_0} + \frac{C_0^\theta}{C_0} \right) \left[ \frac{B_0^2}{A_0^2} \ddot{D} J + \frac{2b}{B_0} I'' - I' \left( \frac{a}{A_0} \right)' \right. \\
& \left. - J'' - \left( J' - \frac{2b}{B_0} I' \right) \left( \frac{A'_0}{A_0} + \frac{C'_0}{C_0} - \frac{B'_0}{B_0} \right) - \frac{1}{r^2} \left( \left( J^\theta - \frac{2b}{B_0} I^\theta \right) \left( \frac{A_0^\theta}{A_0} - \frac{B_0^\theta}{B_0} \right. \right. \right. \\
& \left. \left. \left. + \frac{C_0^\theta}{C_0} \right) - I^\theta \left( \frac{a}{A_0} \right)^\theta \right) \right] + \left( \left( \frac{a}{A_0} \right)^\theta \right. \\
& \left. + 4 \left( \frac{b}{B_0} \right)^\theta \right) \left[ \frac{\alpha R_0^2 B_0^2}{2} + I' \left( \frac{A'_0}{A_0} + \frac{C'_0}{C_0} - \frac{B'_0}{B_0} \right) - \frac{I^\theta}{r^2} \left( \frac{A_0^\theta}{A_0} - \frac{B_0^\theta}{B_0} + \frac{C_0^\theta}{C_0} \right) - I'' \right] \\
& - \frac{C_0^\theta}{C_0} \left[ J'' - \frac{2b}{B_0} I'' + I' \left( \left( \frac{a}{A_0} \right)' - \left( \frac{2b}{B_0} \right)' \right) + \left( J' - \frac{2b}{B_0} I' \right) \left( \frac{A'_0}{A_0} - \frac{B'_0}{B_0} \right. \right. \\
& \left. \left. + \frac{1}{r} \right) + \frac{1}{r^2} \left\{ \left( J^\theta - \frac{2b}{B_0} I^\theta \right) \left( \frac{A_0^\theta}{A_0} - \frac{2B_0^\theta}{B_0} \right) + I^\theta \left( \left( \frac{a}{A_0} \right)^\theta - 2 \left( \frac{b}{B_0} \right)^\theta \right) \right. \right. \\
& \left. \left. + J^{\theta\theta} - \frac{2b}{B_0} I^{\theta\theta} \right\} \right] + \left( \frac{(cC_0)^\theta}{C_0^2} - \frac{2b}{B_0} \frac{C_0^\theta}{C_0} \right) \left[ \frac{\alpha R_0^2 B_0^2}{2} + I' \left( \frac{A'_0}{A_0} - \frac{2B'_0}{B_0} + \frac{1}{r} \right) \right. \\
& \left. + I'' + \frac{I^{\theta\theta}}{r^2} - \frac{I^\theta}{r^2} r \left( \frac{A_0^\theta}{A_0} - \frac{2B_0^\theta}{B_0} \right) \right], \tag{42}
\end{aligned}$$

$$\begin{aligned}
Z_4 = & \frac{A_0^2}{2} \left( \frac{B_0 C_0}{b C_0 - c B_0} \right) \left[ \frac{2}{B_0^2} \left\{ \frac{A'_0 C'_0}{A_0 C_0} \left( \frac{a'}{A'_0} - \frac{a}{A_0} + \frac{c'}{C'_0} - \frac{b}{B_0} \right) + \frac{A''_0}{A_0} \left( \frac{a''}{A''_0} \right. \right. \right. \\
& \left. \left. - \frac{a}{A_0} \right) + \frac{B''_0}{B_0} \left( \frac{b''}{B''_0} - \frac{b}{B_0} \right) + \frac{C''_0}{C_0} \left( \frac{c''}{C''_0} - \frac{b}{B_0} \right) - \frac{1}{r} \left( \frac{a}{A_0} - \frac{2b}{B_0} \right) \right. \\
& \left. \left. - \frac{2B'_0}{B_0} \left( \frac{b}{B_0} \right)' + \frac{2}{r^2} \left\{ \frac{2B_0^\theta}{B_0} \left( \frac{b}{B_0} \right)^\theta + \frac{A_0^{\theta\theta}}{A_0} \left( \frac{a^{\theta\theta}}{A_0^{\theta\theta}} - \frac{a}{A_0} \right) + \frac{B_0^{\theta\theta}}{B_0} \left( \frac{b^{\theta\theta}}{B_0^{\theta\theta}} - \frac{b}{B_0} \right) \right. \right. \right. \\
& \left. \left. \left. + \frac{C_0^{\theta\theta}}{C_0} \left( \frac{c^{\theta\theta}}{C_0^{\theta\theta}} - \frac{b}{B_0} \right) + \frac{A_0^\theta C_0^\theta}{A_0 C_0} \left( \frac{a^\theta}{A_0^\theta} - \frac{a}{A_0} + \frac{c^\theta}{C_0^\theta} - \frac{b}{B_0} \right) \right\} \right\} - e - \frac{2bR_0}{B_0} \right]. \quad (43)
\end{aligned}$$

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